UDC 531

ON THE NONLOCAL APPLICATION OF THE METHOD OF SMALL PARAMETER

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Several applied problems are considered, in which it was possible to obtain a complete picture of the behavior of all bifurcation surfaces without resorting to the method of small parameter, also in the small neighborhood of a conservative system. It was found that in all these problems the bifurcation surfaces associated with the behavior of limit cycles and separatrices, and all their intersection lines (hence, also all possible structures) exist in an as small as desired neighborhood of conservative system sets, and that their mutual position is not related to the closeness to the conservative system. It seems that the realization of that particular situation makes it possible to estimate the behavior of systems that are not too close to conservative by the method of small parameter (a method that is essentially applicable to an uncontrolled small neighborhood of the set of conservative systems).

The possibility of using the qualitative results obtained by the method of small parameter for autonomous systems of second order differential equations for specific parameter values, which in practice is often supported by experiments and is usually considered to be only a plausible assertion [1, 2], since the convergence of series used in it is not known. The idea that obtained results may have real meaning only if the considered series are convergent led to attempts at estimating their radius of convergence [3, 4]. The real advantage of the possibility of nonlocalized application of results obtained by the method of small parameter is, however, different and based on the premise that this method is a device which discloses some features that exist independently of the convergence of series.

1. Let us consider the system

$$\varphi^{\cdot} = y \equiv P, \quad y^{\cdot} = \beta - \sin\varphi - \lambda y - 2\alpha y / (1 + y^2) \equiv Q \quad (1.1)$$

which was investigated in [5, 6] with nonnegative α, β, and λ

We consider the band $-\pi \leqslant \varphi \leqslant \pi$ in the φy -plane with identified edges (the phase system is a cylinder) and assume, for the time being, that $\alpha \geqslant 0$ and $\beta \geqslant 0$.

1.1. The equilibrium state. The saddle parameter. The equilibrium state is on the axis $y = 0$; a saddle exists when $\varphi = \pi - \arcsin\beta$ and a focus (node) when $\varphi = \arcsin\beta$ (when $\lambda = \alpha = 0$) , there is a center). The focus is stable when $\lambda > -2\alpha$ and unstable when $\lambda < -2\alpha$. If $\lambda = -2\alpha$ the focus is complex and the first Liapunov parameter is $\alpha_3 = {}^3/_2\pi\alpha\,(1 - \beta^2)^{1/4} > 0$. Only an unstable limit cycle can contract to the focus or be generated by it.

The saddle parameter ($\Delta \equiv P_\varphi' + Q_y'$ calculated at the saddlepoint) is $\Delta = -(\lambda + 2\alpha)$. This implies that the separatrix loop, if it exists, and a simple focus have the same stability and must be separated by a cycle of opposite stability.

1. 2. Region in which there are no periodic solutions.

By the Bendikson - Dulac criterion there are no cycles and no separatrix loops containing equilibrium states when $\lambda < -2\alpha$ and, also, there are no double cycles that envelop the phase cylinder, since the expression

$$P_\varphi' + Q_y' = -\lambda - 2\alpha\,(1 - y^2) / (1 + y^2)^2$$

does not change its sign.

2°. System (1. 1) is equivalent to the equation

$$y dy + \sin\varphi d\varphi = \beta - \lambda y - 2\alpha y / (1 + y^2)$$

Hence for the closed contour enveloping the cylinder and composed of trajectories of system (1. 1) the following conditions must be satisfied:

$$\int_{-\pi}^{\pi} \left[\beta - \lambda y - \frac{2\alpha y}{(1 + y^2)}\right] d\varphi = 0$$

If, however, $\beta = 0$ and $\lambda < -2\alpha$, the integrand is of constant sign and there are no closed loops enveloping the cylinder.

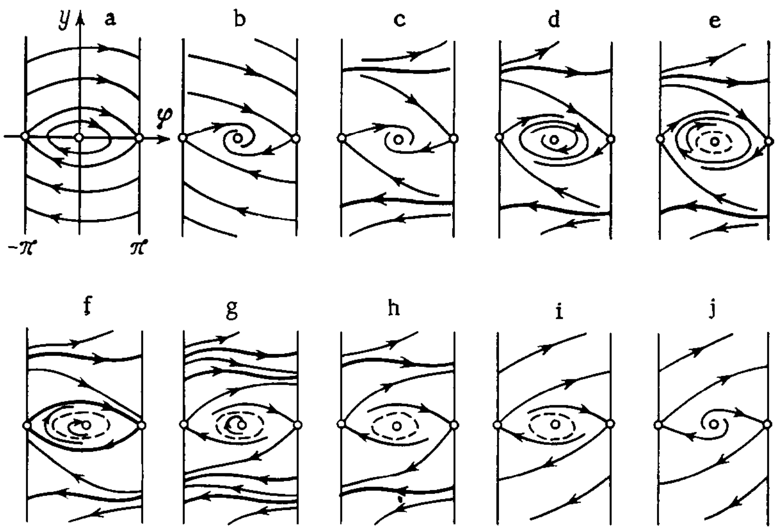


Fig. 1.

1. 3. **Structures with $\beta = 0$.** System (1. 1) is invariant with respect to substitutions $-\varphi$ for φ and $-y$ for y. The phase space is symmetric about the coordinate origin, and when $\lambda = \alpha = 0$ the system is conservative (Fig. 1, a). When α increases the field of directions turns clockwise. Closed curves of a conservative system are without contact. There are no cycles: the α-separatrices run into a stable focus, while the ω-separatrices arrive from infinity (Fig. 1, b). The same structure remains also for all $\lambda > 0$ since with increase of λ the field turns again clockwise. When λ decreases from zero (when $\alpha \neq 0$) an unstable limit cycle (the term $-\lambda y$ which determines the sign of $dy / d\varphi$ for considerable y Fig. 1, c) arrives from infinity.

When $\lambda \leqslant -2\alpha$ the focus is unstable and there are no cycles (see Sect. 1.2) : the ω-separatrices unroll from the focus while the α-separatrices tend to infinity (Fig. 1, j). Since the opposite takes place when $\lambda = 0$ namely, the α-separatrices converge to the focus and the ω-separatrices arrive from infinity, hence when λ changes from 0 to 2α separatrix loops must occur. Owing to symmetry, the loops around the upper and lower half-cylinders occur simultaneously. They can, also, be considered as a single contour enveloping the equilibrium state.

The cycles that arrive from infinity cannot be trapped by separatrix loops, since that is inhibited by the sign of the saddle parameter (only a stable cycle can become a separatrix loop).

The contour consisting of saddles and separatrices which envelops the focus, and the focus itself, are stable, hence they must be separated by an unstable limit cycle (see Sect. 1.1). This implies that, when λ decreases from zero in the interval $-2\alpha < \lambda < 0$, a double limit cycle appears around the equilibrium state formed by the contraction of trajectories. That cycle, owing to the monotonous rotation of the field, separates into two : a stable external and an unstable internal cycle (Fig. 1, e). Then the stable cycle is transformed in a contour consisting of separatrices and saddles (Fig. 1, f). At the disintegration of the contour stable cycles are generated at the upper and lower half-cylinders. Two cycles thus become present on every half-cylinder (Fig. 1, g). With further decrease of λ the cycles on each half-cylinder monotonically converge, then merge (Fig. 1, h) and vanish (Fig. 1, i). When $\lambda = -2\alpha$ the unstable cycle contracts to the focus, yielding the structure (Fig. 1, j without limit cycles. The described bifurcations occur for any $\alpha > 0$ only in the outlined sequence with decreasing λ, hence in the parameter plane $\alpha\lambda$ there exist bifurcation curves lying between the straight lines $\lambda = 0$ and $\lambda = -2\alpha$ which intersect only at the coordinate origin.

1. 4. Structures with $\beta \neq 0$. Let us fix α and consider the plane $\beta\lambda$. Along the straight line $\beta = 0$ for a symmetric phase space bifurcation takes place when λ decreases; when $\lambda = 0$ an unstable limit cycle arrives from infinity, when $\lambda = \lambda_1 < 0$ we have a double limit cycle around the focus formed by trajectory bunching, when $\lambda = \lambda_2 < \lambda_1$ we have a contour formed by separatrices, and when $\lambda = \lambda_3 < \lambda_2$ a double limit cycle formed by the merging of cycles, while for $\lambda = -2\alpha < \lambda_3$ the limit cycle contracts to the focus.

When $\beta > 0$ the phase space symmetry is disturbed and, consequently, the bifurcation curves which correspond to separatrix loops around the lower (curve S_1) and the upper (curve S_2) half-cylinders and, also around the focus (curve S_3) do not coincide in the $\beta\lambda$-plane. They all begin at point $(0, \lambda_2)$ and end on the straight line $\beta = 1$ (when $\beta > 1$ there is no equilibrium state). Curve S_2 has a positive slope, since only when parameters β and λ increase or decrease simultaneously, the nonmonotonous rotation of the field which does not destroy the loop is maintained on the upper half-cylinder. For the same reason curve S_1 has a negative slope. Curve S_3 which lies between S_1 and S_2 and cannot intersect these when $\beta \neq 0$. It ends at point $\lambda = -2\alpha$ of the straight line $\beta = 1$, which corresponds to a degenerate saddle-node without nodal region (the degenerate saddle-node is to be considered as the degeneration of the saddle separatrix which forms a loop when the latter contracts to a point).

The bifurcation curve which corresponds to the double cycle around the focus (curve D) begins at point $(0, \lambda_1)$ and ends at point $(1, -2\alpha)$, where

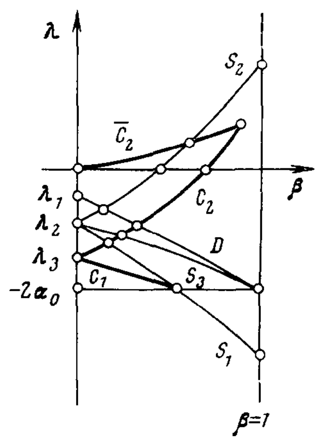


Fig. 2

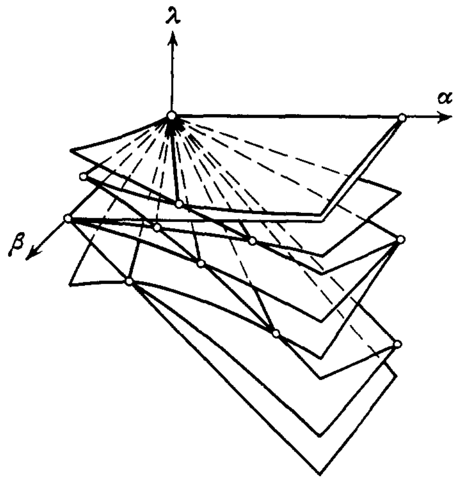


Fig. 3

the saddle parameter vanishes. Curves D and S_3 cannot intersect at any other points.

The bifurcation curves which correspond to double cycles around the lower (curve C_1) and the upper (curve C_2) half-cylinders begin at point $(0, \lambda_3)$. Curve C_1 lies below curve S_1 and has a negative slope; it ends at the point of intersection of the straight line $\lambda = -2\alpha$ on which the saddle parameter Δ vanishes, and curve S_1. Curve C_1 cannot intersect S_1 in the region where $\Delta < 0$, since then a stable loop would be formed, which is impossible owing to the presence of the double cycle on the lower half-cylinder, which is stable from above, moreover it cannot leave region $\Delta < 0$, since no double cycles are possible when $\Delta > 0$ (see Section 1.2). Curve C_2 lies below S_2 and has also a positive slope. It cannot intersect S_2, since that is inhibited by the sign of the saddle parameter (the double cycle on the upper half-cylinder which corresponds to points of curve C_2 is stable from below). Curve C_2 intersects the β-axis at point $\beta = \beta_0 < \alpha$ and passes to region $\lambda > 0$, where it is bounded from above by the straight line $\lambda = \alpha / 4$ (see Appendix 1).

When $\alpha > 0$ curve $\bar{C}_2$ runs from point $(0, 0)$ into region $\beta > 0, \lambda > 0$; it corresponds to the double limit cycle on the upper half-cylinder which is stable from above. That cycle originates at infinity when $\beta = \lambda = 0$ (see Appendix 2).

Curve $\bar{C}_2$ is bounded from above, as curve C_2, by the straight line $\lambda = \alpha / 4$. Both have a positive slope and with increasing β merge at the angle point which corresponds to a triple limit cycle. Curves C_2 and $\bar{C}_2$ must merge since both are bounded from above and from the right (Appendix 3).

The disposition of bifurcation curves in the $\beta\lambda$-plane is shown in Fig. 2 (not to scale) for fixed α. When $\alpha \to 0$ the heavy z-shaped line (curves C_1, C_2, $\bar{C}_2$) contracts to a point. This is accompied by the contraction to point $(0, 0)$ of all other intersection points of bifurcation curves, except the points which lie on the straight line $\beta = 1$. When $\alpha = 0$ curves S_1 and S_2 run from point $(0, 0)$ to points $(1, \pm 1.19)$ [7], and curve S_3 contracts to the segment $\lambda = 0, 0 \leqslant \beta \leqslant 1$, which corresponds to the loop of the conservative system. The partitioning of space $\beta\lambda\alpha$ for positive α and β is shown in Fig. 3. The previously specified constraint on the signs of α and β can now be removed. The bifurcation surfaces can be symmetrically extended with respect to the plane $\beta = 0$ and to axis β. The bifurcations corresponding to points symmetric about the plane $\beta = 0$ appear in the phase space with changed directions of axes φ and y, while those corresponding to points symmetric about the β axis appear with changed direction of the y-axis and reversed direction of motion along trajectories.

1.5. A noteworthy point of the parameter space is that which corresponds to the noncoarse conservative system (the coordinate original). It is the intersection of all bifurcation surfaces and of all lines of their intersections. The analysis of a small neighborhood of that point (by the method of small parameter) provides a comprehensive qualitative picture of the structure and bifurcations with which equilibrium states do not vanish.

Note. All of the above peculiarities of bifurcation surface disposition are obtained also with the more general equation

$$\varphi^{\cdot\cdot} + \lambda\varphi^{\cdot} + F(\varphi) = \beta - \alpha f(\varphi^{\cdot})$$

where function $F(\varphi)$ is continuous, periodic with two extrema in a period, and symmetric about its zeros, while function $f(\varphi^{\cdot})$ is continuous, satisfies conditions $\varphi^{\cdot} f(\varphi^{\cdot}) > 0$ when $\varphi^{\cdot} \neq 0$, $f(\varphi^{\cdot}) = -f(-\varphi^{\cdot})$, $f'(0) > 0$, $f'''(0) < 0$, $f(\infty) = 0$ and is analytic in the neighborhood of $\varphi^{\cdot} = 0$ and $\rho = 1/\varphi^{\cdot} = 0$. The proof is almost verbatim repetition.

2. Let us consider the system

$$x^{\cdot} = y \equiv P, \quad y^{\cdot} = [\varphi(x) - \lambda]\, y + \varphi(x) - \sigma + \alpha y\, (y + 1) \equiv Q \tag{2.1}$$

which on certain assumptions defines the stationary motion of domains in bipolar semiconductors [8-11]. In these equations $\varphi(x)$ is a function with a falling section between ascending branches, and $\lambda > 0$, $\sigma > 0$, $\alpha \geqslant 0$.

2.1. **The equilibrium states.** These are defined by the conditions $y = 0$, $\varphi(x) = \sigma$. Two or three equilibrium states are possible. In the parameter plane $\sigma\lambda$ ($\alpha =$ const) we have a band $\sigma_m < \sigma < \sigma_M$ (σ_m and σ_M are the minimum and maximum values of $\varphi(x)$) in which the system has three equilibrium stated (two saddles and a focus or node between them) and where all bifurcations take place.

The focus is stable when $\lambda > \sigma + \alpha$ and unstable when $\lambda < \sigma + \alpha$. When $\lambda = \sigma + \alpha$ the focus is complex and the first Liapunov parameter is $\alpha_3 = -{}^1/_4\ \alpha\pi\beta^{1/2} < 0$, $\beta = -\varphi'(x_2) > 0$, where x_2 is the coordinate of the complex focus. Hence only a stable limit cycle can contract to, or emanate from the focus.

Since the saddle parameter $\Delta = \sigma - \lambda + \alpha$ and the real part of roots of the characteristic equation for the focus are of the same sign, hence the separatrix loops. if they are formed. and the simple focus are simultaneously either stable or unstable. and must, therefore, be separated by a limit cycle.

2. 2. **The structures of solutions with** $\alpha = 0$. When $\alpha = 0$ and $\sigma = \lambda$, system (2. 1) is conservative and integrable. In the same phase plane there are two saddles with a center between them. When $\sigma \neq \lambda$ system (2. 1) has no closed trajectories, since variation of λ results in a monotonic rotation of the direction field of system (2. 1), and the closed curves of the center are cycles with no contact.

2. 3. **The structures of solutions with** $\alpha \neq 0$. All of the considered bifurcations occur in the band $x_1 < x < x_3$, where x_1 and x_3 are coordinates of the saddle. Because of this, we shall consider only those α- and ω-separatrices which enter that band.

1°. Let $\lambda \gg \sigma$. The equilibrium state is defined by two saddles with a stable node between them. Since the α- and ω- separatrices shift in opposite directions when λ increases, hence the closed trajectories, if they exist, are bounded with respect to y. By applying the Bendikson- Dulac criterion we obtain

$$F_1 \equiv P_x' + Q_y' = \varphi(x) - \lambda + \alpha(2y + 1)$$

When λ is fairly large, the sign of F_1 remains unchanged in the region enveloped by separatrices, where limit cycles could be found, hence there are no cycles in it, the α- separatrices converge to the node and the ω-, separatrices come from infinity (Fig. 5, g).

2°. When $\lambda = \sigma$ the transition from $\alpha = 0$ to $\alpha \neq 0$ results in a turn of the field of system (2. 1) at which the closed curves of the center become loops without contact. Structures with $\alpha = 0$ are shown in Figs. 4, a -4, c, and in Figs. 4, d - 4, f structures with $\alpha > 0$ are plotted. When passing from the structure in Fig. 4, d to that of Fig. 4, e, a separatrix running from saddle to saddle must necessarily appear in region $y < 0$ (Fig. 4, g), and at transition from structure 4, e to structure 4, f this takes place in region $y > 0$ (Fig. 4, h). We denote the values of parameters at which these bifurcations occur by $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$, respectively.

3°. When $\sigma = \text{const}$ and $\alpha = \text{const}$, the monotonic rotation of the direction field of (2. 1) makes it possible to trace all bifurcations produced by the variation of λ When $\lambda = \sigma < \sigma_1$ the ω- separatrix of the right-hand saddle unrolls from the unstable focus and the α-separatrix of the left-hand saddle tends to infinity (Fig. 4, d). When $\lambda \gg \sigma$,, the ω-separatrix of the right-hand saddle comes from infinity and the α- separatrix of the left-hand saddle moves into the stable node (Fig. 5, g). Comparison of the behavior of separatrices shows that when λ increases from $\lambda = \sigma$ to $\lambda \gg \sigma$ bifurcation of separatrices takes place; first, a loop of the right-hand saddle separatrix is formed, and this is followed by the formation of a separatrix running from the saddle to another saddle in region $y > 0$.

Because of the condition $\alpha_3 < 0$ when λ increases and passes through $\lambda = \sigma + \alpha$ a stable cycle contracts to the focus. Since system (2. 1) has no limit cycles when $\lambda = \sigma$ and in the interval of λ variation from σ to $\sigma + \alpha$ only an unstable loop can be formed (the saddle parameter is positive), hence for a certain λ in that interval a semistable cycle is produced by the bunching of trajectories. That cycle is stable from within (Fig. 5, a) which owing to the field monotonic rotation, separates into two :

an unstable external and a stable internal cycles (Fig. 5, b) . With increasing λ the cycles diverge and a separatrix loop is formed, into which during its formation is absorbed an unstable cycle (Fig. 5, c).

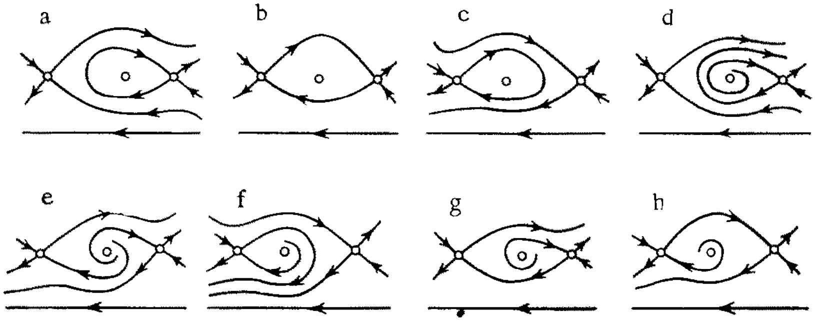


Fig. 4

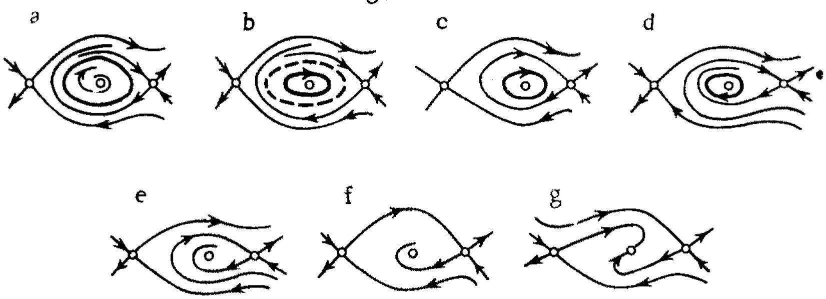
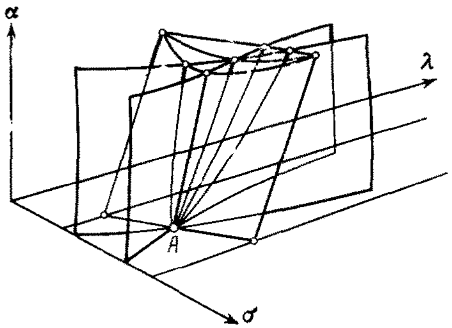


Fig. 5

After disintegration of the loop a structure with one stable limit cycle which envelops the focus, is created (Fig. 5, d). When $\lambda = \sigma + \alpha$ the stable cycle contracts to the focus and for $\lambda > \sigma + \alpha$ we have a structure without limit cycles (Fig. 5, e). Further increase of λ leads to the formation of a separatrix which runs from the saddle to another saddle (Fig. 5, f) at whose disintegration we have a structure that is topologically equivalent to the structure with $\lambda \gg \sigma$ (Fig. 5, g). When λ decreases from $\lambda = \sigma$, the only bifurcation is represented by the separatrix running from the saddle to another saddle in region $y < 0$. Bifurcations can only occur in the indicated order, since the saddle parameter changes its sign together with change of the focus stability.

4°. The sequence of bifurcations (bifurcation curves) in the band $\sigma_2 < \sigma < \sigma_M$ is established in the same way, except that the loop is formed by the separatrix of the left-hand saddle, and that with increasing λ the separatrix running from the saddle to another saddle is generated in region $y < 0$, and when λ decreases this occurs in region $y > 0$.

In the band $\sigma_1 < \upsilon < \sigma_2$ the bifurcations are similar (although some convert in a different order), and the bifurcation curves of adjacent bands convert from one to another. Bifurcation curves of the left-hand and right-hand saddles join at the intersection point of curves that correspond to separatrices running from saddle to saddle, because the separatrix contour with two saddles may be considered as a degenerate right-and left-hand saddles.



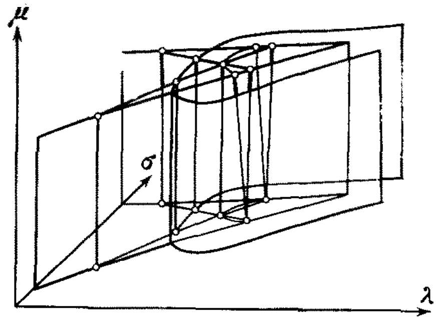


Fig. 6 Fig. 7

A singular point of the kind of saddle-node appears in the phase space on the straight lines $\sigma = \sigma_m$ and $\sigma = \sigma_M$ which bound the considered band in the parameter space. That point degenerates and the nodal region vanishes when a second zero root appears in the solution of the characteristic equation. This occurs at intersections with the straight line $\lambda = \sigma + \alpha$.. The bifurcation curves of separatrix loops and of double cycles also terminate at these points.

2. 4. Subdivision of the parameter space. Subdivision of space $\sigma\lambda\alpha$, is shown in Fig. 6. All possible bifurcation surfaces and their intersection lines, including those corresponding to separatrices running from saddle to saddle, pass through point A which corresponds to the conservative system in the set of conservative systems $\alpha = 0$ and $\lambda = \sigma$ (the possibility of obtaining all bifurcations with limit cycles implies the necessity to have bifurcations of separatrices also in intermediate structures in the same neighborhood). The structures in the neighborhood of point A comprise all qualitative structures and bifurcations of system (2. 1) for parameter values for which equilibrium states do not vanish.

3. Let us consider the system

$$x^{\cdot} = y, \quad y = \sigma - \lambda x - \varphi(x) - \mu y[1 + \varphi'(x)] \qquad (3.1)$$

where $\varphi(x)$ has a dropping section when it is approximated by the cubic polynomial $\varphi(x) = ax^3 - bx^2 + cx$.

The system considered on [12] reduces to the form (3. 1) when $\mu = 1$. For any $\mu > 0$ the equilibrium states, their number, properties and relative position, the equation of the discriminant curve that separates the region of three equilibrium states from that of single equilibrium, and equations of bifurcation curves for the focus are independent of μ. The Liapunov parameters for systems (3. 1) differ from those in [12] by factor μ. All conclusions about the qualitative structures and of the subdivision of parameter space remain unchanged.

When $\mu = 0$ system (3. 1) is conservative and can have either a single equilibrium state, the center, or three, viz. two centers with a saddle between these (for the discriminant curve this is the center and a complex equilibrium state). Using the method of small parameter proposed by Pontriagin [13], it is possible to determine in the plane $\mu = 0$ the disposition of curves which are limiting the bifurcation surfaces when $\mu \to 0$ Bifurcation curves in the parameter space $\lambda\sigma\mu$, their section by the plane $\mu = \text{const}$ and the limit curves are shown in Fig. 7 (bifurcation surfaces related to the complex focus are omitted for simplicity).

The small neighborhood of the conservative system plane comprises all qualitative structures and all bifurcations of system (3.1).

Appendix 1. Existence of the intersection point of curve C_2 with the β-axis and its estimate $(\beta_0 < \alpha)$ follows from the comparison of system (1.1) with $\lambda = 0$ with the system

$$\varphi^{\cdot} = y, \quad y^{\cdot} = \beta - \alpha - \sin\varphi$$

When $\beta > \alpha$ the trajectory of system A. 1) which reach the upper half-cylinder are spirals which tend to infinity (except the ω-separatrix of the saddle, if the latter exists), and the direction field of (1.1) is turned relative to (A. 1) anticlockwise. Because of this system (1.1) has no limit cycles when $\lambda = 0$ and $\beta > \alpha$ which would, however, exist if curve C_2 intersected the β-axis to the right of the considered value $\beta > \alpha$ or tended to infinity with its asymptotic lying below the β-axis.

The boundedness from above is the consequence of the absence of real branches of curve $P_\varphi' + Q_y' = 0$ when $\lambda > \alpha / 4$.

Appendix 2. Setting $y = 1/\rho$ and deriving in the usual way the sequence function in the neighborhood of small $\rho = \rho_0$, we obtain

$$\rho_1(2\pi) - \rho_0(0) = \rho_0^2 \{2\pi\lambda + (4\pi^2\lambda^2 - 2\pi\beta)\rho_0 + [8\pi^3\lambda^3 - 10\pi^2\lambda\beta + 4\pi(\lambda + \alpha) - 6\pi\lambda]\rho_0^2 + \ldots\}$$

When β, γ and $\rho = 1/y$ are small, equation $\rho_1(2\pi) - \rho_0(0) = 0$ has a double root $\rho_0 = \beta / 4\alpha + \ldots$, which corresponds to the limit cycle on the upper half-cylinder, if $\lambda = \beta^2 / 8\alpha + \ldots$. The last expression is the asymptotic representation of curve C_2 in the neighborhood of point $\beta = \lambda = 0$.

Appendix 3. The boundedness of curve C_2 on the right (with respect to β) is implied by that when $\lambda > 0$ it borders on the left on the region of existence of three cycles, while for fairly large (and fixed α) there can be no more than a single cycle. The last follows from the Dulac criterion. Along the straight lines $y = y_j$, where y_j is the root of equation

$$\lambda + 2\alpha(1 - \qquad (1 + y^2)^2 = 0$$

$P_\varphi' + Q_y'$ vanishes and, consequently, λy_j is a bounded quantity. Owing to the boundedness of λy_j we have from (1.1) that for large β $P_\varphi' + Q_y'$ is a curve without contact.

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